Mercati finanziari con N titoli rischiosi

N titoli rischiosi $S_t^1, \dots, S_t^N$ $\quad$ d moti browniani indipendenti $W_t^1, \dots, W_t^d$

$$dS_t^i = S_t^i \left(\mu_i \, dt + \sum_{j=1}^d \sigma_{ij} \, dW_t^j\right) \quad i = 1, \dots, N$$

μ_i rend. atteso $\quad E[S_t^i] = S_0^i e^{\mu_i t}$

$$S_t^i = S_0^i \, e^{\left(\mu_i - \frac{1}{2}\sum_{j=1}^d \sigma_{ij}^2\right)t + \sum_{j=1}^d \sigma_{ij} W_t^j}$$

$\underline{\sigma} = (\sigma_{ij})$. N righe, d colonne

$$E[S_t^i] = S_0^i \, e^{\left(\mu_i - \cancel{\frac{1}{2}\sum_{j=1}^d \sigma_{ij}^2}\right)t} \, \underbrace{E\left[e^{\sum_{j=1}^d \sigma_{ij} W_t^j}\right]}$$

$W_t^j \sim N(0,t)$

$W_t^j \sim \sqrt{t}\, N$

$$= E\left[\prod_{j=1}^d e^{\sigma_{ij} W_t^j}\right] = \prod_{j=1}^d E\left[e^{\sigma_{ij} W_t^j}\right] = \prod_{j=1}^d E\left[e^{\sigma_{ij}\sqrt{t}\, N}\right] = \prod_{j=1}^d e^{\sigma_{ij}^2 t/2}$$

- $e^{x+y} = e^x \cdot e^y$
- $X \perp\!\!\!\perp Y \Rightarrow E[X \cdot Y] = E[X]E[Y]$

$E[e^{aN}] = e^{a^2/2}$

$$= e^{\cancel{\frac{1}{2}\sum_{j=1}^d \sigma_{ij}^2 t}}$$

$E[S_t^i] = S_0^i \, e^{\mu_i t} \quad \Leftarrow$

μ_i rend. atteso dell'azione

Def. Q misura mg per il m.f. se $\tilde{S}^i_t = S^i_t e^{-rt}$ $\forall i = 1, \dots, N$ sono Q-mg (S^i_t ha rend. atteso r rispetto a Q)

$Q \sim P$

$$dS^i_t = S^i_t \left(\mu_i dt + \sum_{j=1}^{d} \sigma_{ij} dW^j_t\right)$$

$$\downarrow$$

$$dS^i_t = S^i_t \left(r\, dt + \sum_{j=1}^{d} \sigma_{ij} dW^{\theta,j}_t\right)$$

$W^{\theta,j}_t = \theta_j t + W^j_t$ $\quad Q^\theta$- moti browniani

Q^θ è associata alla soluzione θ del sistema lineare

$$\underline{\sigma} \cdot \underline{\theta} = \begin{pmatrix} \mu_1 - r \\ \vdots \\ \mu_N - r \end{pmatrix} = \underline{\mu} - \underline{r}$$

$N \times d \quad d \times 1 \quad N \times 1$

prodotto righe per colonne

$\theta = \begin{pmatrix} \theta_1 \\ \vdots \\ \theta_d \end{pmatrix}$ vettore prezzo di mercato del rischio

$$\tilde{S}^i_t = S^i_t e^{-rt} = e^{-\frac{1}{2}\sum_{j=1}^{d} \sigma^2_{ij} t + \sum_{j=1}^{d} \sigma_{ij} W^{\theta,j}_t} \Big\} \; Q^\theta\text{- mg} \quad \left(e^{-\frac{1}{2}\sigma^2 t + \sigma W_t}\right)$$

I Teorema fondamentale dell'Asset Pricing

Un mercato è libero da arbitraggi $\Leftrightarrow$ $\exists$ almeno una misura mg equivalente

II Teorema fondamentale dell'Asset Pricing

Un mercato libero da arbitraggi è completo $\Leftrightarrow$ $\exists!$ (ed è unica) Q mis. mg eq.

Ogni derivato può essere replicato da una strategia autofinanziante

Sistema lineare $\underline{\sigma} \cdot \underline{\theta} = \underline{\mu} - \underline{r}$

$N \times d \quad d \times 1 \quad N \times 1$

Mercato libero da arbitraggi e completo

- Se $N = d$ se $\det \underline{\sigma} \neq 0$ $\exists!\ \underline{\theta} = \underline{\sigma}^{-1}(\underline{\mu} - \underline{r})$ Si usa teorema di Cramér

$$\theta_i = \frac{\begin{vmatrix} \sigma_{11} \cdots \mu_1 - r & \sigma_{1d} \\ \vdots & \vdots \\ \sigma_{N1} \cdots \mu_N - r \cdots & \sigma_{Nd} \end{vmatrix}}{\det \underline{\sigma}}$$

Mercati incompleti

- Se $N < d$ il teorema di Rouché-Capelli il sistema ammette sol. $\Leftrightarrow$ $rg(\underline{\sigma}) = rg(\underline{\sigma}, \underline{\mu} - \underline{r})$ $= K$

osserviamo che $K < d$ e ne ammette ∞^{d-k} sono infinite

- Se $N > d$ se $\boxed{rg(\underline{\sigma}) = rg(\underline{\sigma}, \underline{\mu} - \underline{r}) = K}$ per R-C.

$$\begin{cases} \text{se } K = d & \text{una sola soluzione} \\ \text{se } K < d & \infty^{d-k} \text{ sol.} \end{cases}$$

$\exists!\ Q$ M. compl.

$\exists\ \infty\ Q$ M. incompleto

Esempio $N=2$ $d=1$

$$\begin{cases} dS_t^1 = S_t^1(\mu_1\,dt + \sigma_1\,dW_t) \\ dS_t^2 = S_t^2(\mu_2\,dt + \sigma_2\,dW_t) \end{cases}$$

$$\underset{2\times 1}{\begin{pmatrix}\sigma_1\\ \sigma_2\end{pmatrix}} \cdot \underset{1\times 1}{\theta} = \underset{2\times 1}{\begin{pmatrix}\mu_1 - r\\ \mu_2 - r\end{pmatrix}} \qquad \sigma_1, \sigma_2 > 0$$

$$\begin{cases}\sigma_1\theta = \mu_1 - r\\ \sigma_2\theta = \mu_2 - r\end{cases} \qquad \begin{cases}\theta = \dfrac{\mu_1-r}{\sigma_1}\\ \theta = \dfrac{\mu_2 - r}{\sigma_2}\end{cases} \Rightarrow$$

1) se $\dfrac{\mu_1 - r}{\sigma_1} = \dfrac{\mu_2 - r}{\sigma_2}$ $\exists!\ \theta = \ldots =$

$\exists!\ Q^\theta$ mercato libero da arb. e completo

2) se $\dfrac{\mu_1-r}{\sigma_1} \neq \dfrac{\mu_2 - r}{\sigma_2}$

ie sistema non ammette soluzioni

$\nexists Q$ ie mercato ammette arbitraggi

2) $rg\,\underline{\sigma} = 1$

$$\begin{vmatrix}\sigma_1 & \mu_1 - r\\ \sigma_2 & \mu_2 - r\end{vmatrix} = \sigma_1(\mu_2 - r) - \sigma_2(\mu_1 - r) \neq 0$$

$$rg(\underline{\sigma}, \underline{\mu} - \underline{r}) = 2$$

e ora costruiamo un arbitraggio

Se $\lambda = \dfrac{\mu_1 - r}{\sigma_1} - \dfrac{\mu_2 - r}{\sigma_2} > 0$ $\quad \alpha_t^1 = \dfrac{1}{S_t^1\sigma_1}$ $\quad \alpha_t^2 = -\dfrac{1}{S_t^2\sigma_2}$

$\alpha_t^1 > 0$ acq. l'azione S^1

$\alpha_t^2 < 0$ vendo allo scop. S^2

Valore del portafoglio al tempo t

$$V_t = \alpha_t^1 S_t^1 + \alpha_t^2 S_t^2 + \beta_t B_t \qquad \beta_t = \frac{V_t - (\alpha_t^1 S_t^1 + \alpha_t^2 S_t^2)}{B_t}$$

autofinanziante

$$\begin{cases} dB_t = r B_t\,dt\\ B_t = e^{rt}\\ B_0 = 1\end{cases}$$

$$dV_t = \alpha_t^1\,dS_t^1 + \alpha_t^2\,dS_t^2 + \beta_t\,\underbrace{dB_t}_{rB_t\,dt}$$

$$dV_t = \alpha^1_t \, dS^1_t + \alpha^2_t \, dS^2_t + \frac{(V_t - \alpha^1_t S^1_t - \alpha^2_t S^2_t)}{\cancel{B_t}}\, r \cancel{B_t} \, dt \qquad (\alpha^1, \alpha^2, \beta)$$

$$\alpha^1_t = \frac{1}{S^1_t \sigma_1} \qquad \alpha^2_t = -\frac{1}{S^2_t \sigma_2}$$

$$dV_t = \frac{1}{\cancel{S^1_t}\sigma_1} \cancel{S^1_t} \{\mu_1 dt + \sigma_1 dW_t\} - \frac{1}{\cancel{S^2_t}\sigma_2} \cancel{S^2_t} \{\mu_2 dt + \sigma_2 dW_t\}$$

$$+ r\left\{ V_t - \frac{1}{\sigma_1} + \frac{1}{\sigma_2} \right\} dt = \frac{\mu_1}{\sigma_1} dt + \cancel{dW_t} - \frac{\mu_2}{\sigma_2} dt - \cancel{dW_t} + r V_t \, dt$$

$$- \frac{r}{\sigma_1} dt + \frac{r}{\sigma_2} dt$$

$$dV_t = \Big\{ \underbrace{\frac{\mu_1 - r}{\sigma_1} - \frac{\mu_2 - r}{\sigma_2}}_{>0} \Big\} dt + r V_t \, dt$$

Questo portafoglio è un arbitraggio perché è privo di rischio ma ha un rendimento $> r$

- $\begin{cases} dS^1_t = S^1_t(\mu_1 dt + \sigma_1 dW^1_t) \\ dS^2_t = S^2_t(\mu_2 dt + \sigma_2 dW^2_t) \end{cases}$ $\quad \underline{\theta} \begin{cases} \theta_1 = \frac{\mu_1 - r}{\sigma_1} \\ \theta_2 = \frac{\mu_2 - r}{\sigma_2} \end{cases}$ $\quad \exists!\, Q$

W^1 e W^2 mot. brow. indip.

- $\begin{cases} dS^1_t = S^1_t(\mu_1 dt + \sigma_1 dW^1_t) \\ dS^2_t = S^2_t(\mu_2 dt + \sigma_2 dZ_t) \end{cases}$

W^1_t e Z_t moti browniani correlati

$$\boxed{\mathbb{E}[W^1_t Z_t] = \rho t}$$

$\rho \neq 0$

$\rho \in (-1, 1)$

$X, Y \quad \rho = \frac{Cov(X,Y)}{\sqrt{Var(X)}\sqrt{Var(Y)}} = \frac{\mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)}{\sqrt{Var(X)}\sqrt{Var(Y)}}$

$X \sim N(0,t)$

$Y \sim N(0,t)$

$$\rho = \frac{\mathbb{E}[XY]}{\sqrt{t}\sqrt{t}} \qquad \mathbb{E}[XY] = \rho \cdot t$$

Osserviamo che $\boxed{Z_t = \rho W^1_t + \sqrt{1-\rho^2}\, W^2_t}$

W^1_t, W^2_t moti browniani indipendenti

$Var(aX) = a^2 Var(X)$

$\mathbb{E}[Z_t] = 0 \qquad Var(Z_t) = Var(\rho W^1_t) + Var(\sqrt{1-\rho^2}\, W^2_t)$

$$= \rho^2 \underbrace{Var(W^1_t)}_{= t} + (1-\rho^2) \underbrace{Var(W^2_t)}_{= t} = t$$

per le prob. delle gaussiane $Z_t \sim N(0,t)$

$$\mathbb{E}[Z_t W^1_t] = \mathbb{E}[\rho \cdot (W^1_t)^2] + \mathbb{E}[\sqrt{1-\rho^2}\, W^1_t \cdot W^2_t] = \rho t$$

$$\rho \underbrace{\mathbb{E}[(W^1_t)^2]}_{= t} + \sqrt{1-\rho^2}\, \underbrace{\mathbb{E}(W^1_t)\mathbb{E}(W^2_t)}_{= 0}$$

$$\begin{cases} dS_t^1 = S_t^1(\mu_1\,dt + \sigma_1\,dW_t^1) \\ dS_t^2 = S_t^2(\mu_2\,dt + \sigma_2\cdot\rho\,dW_t^1 + \sigma_2\sqrt{1-\rho^2}\,dW_t^2) \end{cases}$$

W^1 e W^2 moti brow. ind. $\sigma_1,\sigma_2>0$

$\rho\in(-1,1)$

$$\underline{\sigma} = \begin{pmatrix} \sigma_1 & 0 \\ \sigma_2\rho & \sigma_2\sqrt{1-\rho^2} \end{pmatrix} \qquad \underline{\sigma}\begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix} = \begin{pmatrix}\mu_1 - r\\ \mu_2 - r\end{pmatrix}$$

$$\begin{cases} \sigma_1\theta_1 = \mu_1 - r \\ \sigma_2\rho\,\theta_1 + \sigma_2\sqrt{1-\rho^2}\,\theta_2 = \mu_2 - r \end{cases} \qquad \theta_1 = \frac{\mu_1 - r}{\sigma_1}$$

$\operatorname{Det}\underline{\sigma} = \sigma_1\cdot\sigma_2\sqrt{1-\rho^2} > 0$, $\neq 0$

$\exists!\ \underline{\theta} = (\theta_1,\theta_2)$

$$\sigma_2\sqrt{1-\rho^2}\,\theta_2 = \mu_2 - r - \sigma_2\rho\cdot\frac{\mu_1 - r}{\sigma_1} \qquad \theta_2 = \frac{1}{\sqrt{1-\rho^2}}\,\frac{\mu_2 - r}{\sigma_2} - \frac{\rho}{\sqrt{1-\rho^2}}\,\frac{\mu_1 - r}{\sigma_1}$$

$\exists!\ \theta$ per i teoremi fond. dell'asset pricing $\quad Q^\theta$

il mercato non ammette arbitraggi ed è completo

in senso Q^θ:

$$\begin{cases} dS_t^1 = S_t^1(r\,dt + \sigma_1\,dW_t^{1,\theta}) \\ dS_t^2 = S_t^2(r\,dt + \underbrace{\sigma_2\cdot\rho\,dW_t^{1,\theta} + \sigma_2\sqrt{1-\rho^2}\,dW_t^{2,\theta}}_{\sigma_2\,Z_t^\theta}) \end{cases}$$

$W_t^{1,\theta}$ e $W_t^{2,\theta}$ Q^θ-moti brow ind.

Z_t^θ Q^θ-mot. b.

$\mathbb{E}[Z_t^\theta\cdot W_t^1] = \rho t$

Derivato $F(S_T^1, S_T^2) = S_T^1 \sqrt{S_T^2}$

$$f(t,x_1,x_2) = e^{-r(T-t)}\,\mathbb{E}^Q\left[S_T^1 \sqrt{S_T^2} \mid S_t^1 = x_1,\ S_t^2 = x_2 \right]$$

$$S_T^1 = S_t^1\, e^{(r-\sigma_1^2/2)(T-t) + \sigma_1 (W_T^{1Q} - W_t^{1Q})}$$

$$S_T^2 = S_t^2\, e^{(r-\sigma_2^2/2)(T-t) + \sigma_2 (Z_T^Q - Z_t^Q)} \qquad Z_t = \rho W_t^1 + \sqrt{1-\rho^2}\, W_t^2$$

$W_T^{1Q} - W_t^{1Q}$

$$= e^{-r(T-t)} x_1 e^{(r-\sigma_1^2/2)(T-t)}\, \mathbb{E}^Q\left[e^{\sigma_1 (W_T^{1Q} - W_t^{1Q})} \cdot \sqrt{S_t^2}\, e^{\frac{1}{2}(r-\sigma_2^2/2)(T-t) + \frac{1}{2}\sigma_2(\rho W_t^1 + \sqrt{1-\rho^2}\, W_t^2)} \mid S_t^2 = x_2 \right]$$

$W_T^{2Q} - W_t^{2Q}$

$$= \cancel{e^{-r(T-t)}} x_1 e^{(r-\cancel{\sigma_1^2/2})(T-t)} \sqrt{x_2}\, e^{\frac{1}{2}(r-\sigma_2^2/2)(T-t)} \cdot$$

$$\cdot\ \underbrace{\mathbb{E}^Q\left[e^{\sigma_1 (W_T^{1Q} - W_t^{1Q}) + \frac{1}{2}\sigma_2 \rho (W_T^{1Q} - W_t^{1Q})} \cdot e^{\frac{1}{2}\sigma_2 \sqrt{1-\rho^2}(W_T^{2Q} - W_t^{2Q})} \right]}$$

perché $W^{1,Q} \perp W^{2,Q}$

$$\mathbb{E}^Q\left[e^{(\sigma_1 + \frac{1}{2}\sigma_2 \rho)\underbrace{(W_T^{1Q} - W_t^{1Q})}_{\sqrt{T-t}\,N}} \right] \mathbb{E}^Q\left[e^{\frac{1}{2}\sigma_2 \sqrt{1-\rho^2}\underbrace{(W_T^{2Q} - W_t^{2Q})}_{\sqrt{T-t}\,N}} \right]$$

$$\mathbb{E}[e^{aN}] = e^{a^2/2}$$

$$\underbrace{e^{\frac{(\sigma_1 + \frac{1}{2}\sigma_2\rho)^2 (T-t)}{2}}}_{e^{\cancel{\frac{\sigma_1^2}{2}(T-t)} + \frac{\sigma_1\sigma_2\rho}{2}(T-t) + \cancel{\frac{1}{8}\sigma_2^2\rho^2(T-t)}}} \cdot e^{\frac{1}{8}\sigma_2^2(1-\cancel{\rho^2})(T-t)} = x_1\sqrt{x_2}\, e^{\frac{1}{2}r(T-t)}\, e^{-\frac{1}{4}\sigma_2^2(T-t)}$$

$$\cdot\ e^{\frac{1}{2}\rho\sigma_1\sigma_2(T-t)}\, e^{\frac{1}{8}\sigma_2^2(T-t)}$$

$$-\frac{1}{4} + \frac{1}{8} = -\frac{1}{8}$$

$$f(t,x_1,x_2) = x_1\sqrt{x_2}\, e^{\frac{1}{2}r(T-t)}\, e^{\frac{1}{2}\rho\sigma_1\sigma_2(T-t)}\, e^{-\frac{1}{8}\sigma_2^2(T-t)}$$

Strategia Delta-hedging $\quad f(t,x_1,x_2) = x_1 \sqrt{x_2}\, e^{c(T-t)}$

$\alpha_t^1 = \frac{\partial f}{\partial x_1} = \sqrt{x_2}\, e^{c(T-t)}$ quote in S^1

$\alpha_t^2 = \frac{\partial f}{\partial x_2} = \frac{x_1}{2\sqrt{x_2}}\, e^{c(T-t)}$ quota in S^2

- $V(t,x_1,x_2) = \underbrace{-f(t,x_1,x_2)}^{\text{corto sul derivato}} + \alpha^1 x_1 + \alpha^2 x_2$

$\frac{\partial V}{\partial x_1} = \frac{\partial V}{\partial x_2} = 0 \Rightarrow \alpha^1 = \frac{\partial f}{\partial x_1} \quad \alpha^2 = \frac{\partial f}{\partial x_2}$

Delta neutrale

Approfondimento: Eq. di valutazione $f(t,x_1,x_2)$

$$\begin{cases} \frac{\partial f}{\partial t} + r x_1 \frac{\partial f}{\partial x_1} + \frac{1}{2}\sigma_1^2 x_1^2 \frac{\partial^2 f}{\partial x_1^2} + r x_2 \frac{\partial f}{\partial x_2} + \frac{1}{2}\sigma_2^2 x_2^2 \frac{\partial^2 f}{\partial x_2^2} + \rho\sigma_1\sigma_2 \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \quad - r f = 0 \\ f(T,x_1,x_2) = F(x_1,x_2) \end{cases}$$