Proprietà di Markov

$(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{t\geq 0})$ $\quad \mathcal{F}_t \subseteq \mathcal{F}_s \subseteq \mathcal{F} \quad \forall t \leq s$

(1) $\boxed{P(X_t \in I \mid \mathcal{F}_s) = P(X_t \in I \mid X_s) \quad \forall t > s}$ $\quad I \subseteq \mathbb{R}$, $I \in \mathcal{B}(\mathbb{R})$

$\mathcal{F}_s$: passato fino al tempo s

X_s: presente

• Condizionatamente al presente il futuro è indipendente dal passato.

$\forall f(x)$

(2) $\mathbb{E}[f(X_t) \mid \mathcal{F}_s] = \mathbb{E}[f(X_t) \mid X_s]$ $\quad f(x) = \mathbb{1}_I(x)$

• $\mathcal{B} \subseteq \mathcal{F}$ sotto sigma-algebre

$X \in \mathcal{B}$-mis $\quad \mathbb{E}(X \mid \mathcal{B}) = X$

$Y \perp\!\!\!\perp \mathcal{B} \quad \mathbb{E}(Y \mid \mathcal{B}) = \mathbb{E}(Y)$

$\mathbb{E}(XY \mid \mathcal{B}) = X\,\mathbb{E}(Y \mid \mathcal{B}) = X\,\mathbb{E}(Y)$

$$\mathbb{E}[\underbrace{f(X)}_{\mathcal{B}\text{-mis.}}\overbrace{g(Y)}^{\perp\!\!\!\perp \mathcal{B}} \mid \mathcal{B}] = f(X)\,\mathbb{E}[g(Y)]$$

$h(x,y) = f(x)\,g(y)$

$\Downarrow$

<u>Lemma</u>: $h(x,y)$ $\quad \mathbb{E}[h(X,Y) \mid \mathcal{B}] = \mathbb{E}[h(x,Y)]\big|_{x=X}$

Proposizione

Il moto browniano è un processo di Markov

Dim. $\forall f(x) \quad t \geq s$

$$\mathbb{E}[f(W_t) \mid \mathcal{F}_s] = \mathbb{E}[f(\underbrace{W_t - W_s}_{\perp \mathcal{F}_s \ \to\ Y} + \underbrace{W_s}_{\mathcal{F}_s\text{-misurabile}\ \to\ X}) \mid \mathcal{F}_s]$$

$\to X$, $\to \mathcal{B}$

- Il moto browniano ha incrementi indipendenti $W_t - W_s \perp\!\!\!\perp \mathcal{F}_s$

per il Lemma precedente

$$= \underbrace{\mathbb{E}[f(W_t - W_s + x)]}_{g(x)}\Big|_{x = W_s} = g(W_s)$$

- Prob. delle torre $\mathcal{B}_1 \subseteq \mathcal{B}_2$

$$\mathbb{E}(\mathbb{E}(X \mid \mathcal{B}_1) \mid \mathcal{B}_2) = \mathbb{E}(\mathbb{E}(X \mid \mathcal{B}_2) \mid \mathcal{B}_1) = \mathbb{E}(X \mid \mathcal{B}_1)$$

$$\Rightarrow \quad \mathbb{E}[f(W_t) \mid \mathcal{F}_s] = \mathbb{E}[f(W_t) \mid W_s]$$

perché

$$\underbrace{\mathbb{E}[f(W_t) \mid W_s]}_{\sigma(W_s) \subseteq \mathcal{F}_s} = \mathbb{E}[\underbrace{\mathbb{E}[f(W_t) \mid \mathcal{F}_s]}_{g(W_s) \text{ è } \sigma(W_s)\text{-mis.}} \mid W_s] = g(W_s)$$

$$\| $$

$$\mathbb{E}[f(W_t) \mid \mathcal{F}_s] \qquad \square$$

Oss:

$$g(x) = \mathbb{E}[f(\overbrace{W_t - W_s}^{\sim \mathcal{N}(0, t-s)} + x)] = \int_{\mathbb{R}} f(y+x) \underbrace{\frac{1}{\sqrt{2\pi(t-s)}} e^{-\frac{y^2}{2(t-s)}}}_{\text{densità di } W_t - W_s} dy$$

Proposizione

Ogni p. ad incrementi indipendenti gode della p. di Markov

Prop. $S_t = S_0 e^{(\mu - \sigma^2/2)t + \sigma W_t}$ p. dei prezzi di un'azione nel modello di B&S

$\{S_t\}_{t \geq 0}$ è un p. di Markov

Dim. $S_t = f(t, W_t)$ $f(t,x) = S_0 e^{(\mu - \sigma^2/2)t + \sigma x}$

Coerente con l'efficienza debole

dobbiamo mostrare che $\forall g$ $\mathbb{E}[g(S_t) | \mathcal{F}_s] = \mathbb{E}[g(S_t) | S_s]$ $\forall t \geq s$

$\mathbb{E}[g(S_t) | \mathcal{F}_s] = \mathbb{E}[g(f(t, W_t)) | \mathcal{F}_s] = \mathbb{E}[g(f(t, W_t)) | W_s]$

$F = g \circ f$ W è di Markov

$= \mathbb{E}[g(S_t) | W_s] = \mathbb{E}[g(S_t) | S_s]$

essendo $\sigma(W_s) = \sigma(S_s)$

$\square$

Poiché $S_t = f(t, W_t) \Rightarrow S_t$ è $\sigma(W_t)$-mis.

$$\boxed{Z = f(X) \Rightarrow Z \text{ è } \sigma(X)\text{-mis}}$$

$$S_t = S_0 e^{(\mu - \frac{\sigma^2}{2})t + \sigma W_t}$$

$$\frac{S_t}{S_0} = e^{(\mu - \frac{\sigma^2}{2})t + \sigma W_t}$$

prendo $\lg$ ...

$$\lg\left(\frac{S_t}{S_0}\right) = (\mu - \tfrac{\sigma^2}{2})t + \sigma W_t \Rightarrow W_t = \frac{1}{\sigma}\left\{\lg\left(\frac{S_t}{S_0}\right) - (\mu - \tfrac{\sigma^2}{2})t\right\}$$

$$W_t = h(t, S_t)$$

$$\Downarrow$$

W_t è $\sigma(S_t)$-mis.

$$\sigma(W_t) = \sigma(S_t)$$

Le Martingale

Un p.s. $\{M_t\}_{t\geq 0}$ è una mg se

$$\forall\ t \geq s \qquad \mathbb{E}(M_t \mid \mathcal{F}_s) = M_s$$

$$\mathbb{E}(\mathbb{E}(X \mid \mathcal{B})) = \mathbb{E}(X)$$

$$\Rightarrow \mathbb{E}(\mathbb{E}(M_t \mid \mathcal{F}_s)) = \mathbb{E}(M_t) = \mathbb{E}(M_s) \quad \forall\ t \geq s$$

$$\mathbb{E}(M_s) = \mathbb{E}(M_0) = M_0 \qquad M_0 \in \mathbb{R}$$

le martingale hanno media costante

Esempi

1) mg regolare $\qquad M_t = \mathbb{E}(X \mid \mathcal{F}_t) \qquad X$ v.a.

$$\forall\ t \geq s \quad \mathbb{E}(M_t \mid \mathcal{F}_s) = \mathbb{E}(\mathbb{E}(X \mid \mathcal{F}_t) \mid \mathcal{F}_s) = \mathbb{E}(X \mid \mathcal{F}_s) = M_s$$

$\mathcal{F}_s \subseteq \mathcal{F}_t \quad$ p. della torre

In particolare

$$\mathbb{E}(M_t) = \mathbb{E}(\mathbb{E}(X \mid \mathcal{F}_t)) = \mathbb{E}(X) \quad \forall\ t \geq 0$$

2) Il moto browniano è una martingala

$$E[W_t \mid \mathcal{F}_s] = W_s \qquad \forall\, t > s$$

infatti $E[W_t \mid \mathcal{F}_s] = E[W_t - W_s + W_s \mid \mathcal{F}_s] = E[\overbrace{W_t - W_s}^{\perp\!\!\!\perp \mathcal{F}_s} \mid \mathcal{F}_s] + E[\overbrace{W_s}^{\mathcal{F}_s\text{-mis.}} \mid \mathcal{F}_s]$

$= \underbrace{E[W_t - W_s]}_{=0} + W_s = W_s$

$W_t - W_s \sim \mathcal{N}(0, t-s)$ — 0: media, $t-s$: varianza

$E(W_t - W_s) = 0$

$W_t \sim \mathcal{N}(0, t)$

- Ogni processo ad incrementi indipendenti e media costante è una mg

$X_t - X_s \perp\!\!\!\perp \mathcal{F}_s \qquad E(X_t) = E(X_s) \quad \forall\, t \geq s \implies \{X_t\}$ è una mg

3) $M_t = W_t^2 - t$ è una mg

$E(M_t \mid \mathcal{F}_s) = E(W_t^2 - t \mid \mathcal{F}_s) = E((\overbrace{W_t - W_s}^{a} + \overbrace{W_s}^{b})^2 - t \mid \mathcal{F}_s) =$

$= E[(W_t - W_s)^2 + W_s^2 + 2W_s(W_t - W_s) - t \mid \mathcal{F}_s] =$

$= E[\underbrace{(W_t - W_s)^2}_{\perp\!\!\!\perp \mathcal{F}_s} \mid \mathcal{F}_s] + E[\underbrace{W_s^2}_{\text{è } \mathcal{F}_s\text{-mis.}} \mid \mathcal{F}_s] + 2\,E[\underbrace{W_s}_{\mathcal{F}_s\text{-mis.}}(W_t - W_s) \mid \mathcal{F}_s] - t$

$$\underbrace{\mathbb{E}[(W_t-W_s)^2]}_{t-s} + W_s^2 + 2W_s\underbrace{\mathbb{E}[W_t-W_s]}_{=0} - t$$

X è $\mathcal{B}$-mis

$\mathbb{E}(X|\mathcal{B}) = X$

$\mathbb{E}(XY|\mathcal{B}) = X\mathbb{E}(Y|\mathcal{B})$

$Y \perp\!\!\!\perp \mathcal{B}$

$\mathbb{E}(Y|\mathcal{B}) = \mathbb{E}(Y)$

$\boxed{\mathbb{E}[e^{aN}] = e^{a^2/2} \quad a\in\mathbb{R}}$

$W_t - W_s \sim \mathcal{N}(0, t-s)$

$\mathrm{Var}(W_t - W_s) = \mathbb{E}[(W_t-W_s)^2] = t-s$

$$= t-s+W_s^2 - t = W_s^2 - s$$

$\mathrm{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2$

ossia $M_t = W_t^2 - t$ è una mg $\quad \mathbb{E}(M_t) = \mathbb{E}(W_t^2 - t) = 0$

Martingala Esponenziale

4) $Z_t = e^{-\frac{1}{2}c^2 t + cW_t}$ $\quad c\in\mathbb{R}$ è una mg $\quad \mathbb{E}(Z_t|\mathcal{F}_s) = Z_s \quad \forall t>s$

$$\mathbb{E}(Z_t|\mathcal{F}_s) = \mathbb{E}(\underbrace{e^{-\frac{1}{2}c^2t + cW_t}}_{e^{-\frac{1}{2}c^2t}\, e^{cW_t}}|\mathcal{F}_s) = e^{-\frac{1}{2}c^2t}\,\mathbb{E}(\underbrace{e^{c(W_t-W_s+W_s)}}_{e^{cW_s}\cdot e^{c(W_t-W_s)}}|\mathcal{F}_s) =$$

e^{cW_s}: $\mathcal{F}_s$-mis. $\quad$ $W_t - W_s \perp\!\!\!\perp \mathcal{F}_s$

$$= e^{-\frac{1}{2}c^2t}\, e^{cW_s}\, \mathbb{E}(e^{c(W_t-W_s)}) = e^{-\frac{1}{2}c^2t}\, e^{cW_s}\, \underbrace{\mathbb{E}[e^{c\sqrt{t-s}\,N}]}_{e^{\frac{c^2}{2}(t-s)}} = e^{-\frac{1}{2}c^2 s + cW_s} = Z_s$$

$W_t - W_s \sim \sqrt{t-s}\, N \qquad N \sim \mathcal{N}(0,1)$

Z_t è una mg $\Rightarrow$ $\mathbb{E}(Z_t) = Z_0 = e^{-\frac{1}{2}c^2 0 + c \overset{=0}{W_0}} = e^0 = 1$

ha media 1

- $S_t = S_0 \, e^{(\mu - \frac{\sigma^2}{2})t + \sigma W_t} = S_0 \, e^{\mu t} \, \underbrace{e^{-\frac{1}{2}\sigma^2 t + \sigma W_t}}_{Z_t}$

Z_t è una mg

$\mathbb{E}(Z_t) = 1$

$\Rightarrow \mathbb{E}(S_t) = S_0 \, e^{\mu t} \, \underbrace{\mathbb{E}(Z_t)}_{= 1} = S_0 \, e^{\mu t}$

Foglio 2

es 2) $\sigma = 30\%$ $\quad \mu = 15\%$ $\quad$ base annue

$S_0 = 100$ € $\quad$ 1 mese

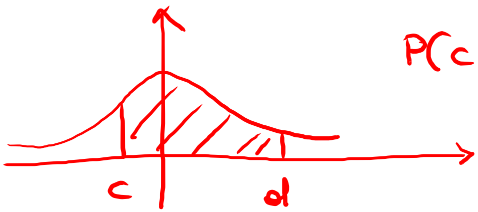
$t = \frac{1}{12}$ anno

$$S_t = S_0 \, e^{(\mu - \frac{\sigma^2}{2})t + \sigma W_t} \qquad P(90 \le S_t \le 110) = ?$$

$$P(a \le S_t \le b) = P(a \le S_0 e^{\overline{\quad}} \le b) = P\left(\frac{a}{S_0} \le e^{\overline{\quad}} \le \frac{b}{S_0}\right)$$

prendo il log...

$W_t \sim \sqrt{t}\, N \qquad N \sim \mathcal{N}(0,1)$ $\quad$ <u>TAVOLE</u>



$$P(c \le N \le d) = \Phi(d) - \Phi(c)$$

$$\Phi(x) = P(N \le x)$$

$$\mathbb{E}(M_t \mid \mathcal{F}_s) \overset{\geq}{\underset{\leq}{=}} M_s \quad \forall t > s$$

submg

super mg

- M_t è una submg $\quad \underbrace{\mathbb{E}(\mathbb{E}(M_t \mid \mathcal{F}_s))}_{\mathbb{E}(M_t)} \geq \mathbb{E}(M_s)$

  $\mathbb{E}(M_t) \geq \mathbb{E}(M_s) \geq M_0$ $\quad$ medie crescente nel tempo

  $t > s$

- M_t è una supermg $\Rightarrow$ medie decrescente nel tempo

  $\mathbb{E}(M_t) \leq \mathbb{E}(M_s) \leq M_0 \quad \forall t > s$