

2) Probabilită e medic condiționată de una τ -algebra

PROBABILITÀ CONDIZIONATA AD UNA σ -ALGEBRA

$(\Omega, \mathcal{F}, P)$

$\mathcal{G} \subseteq \mathcal{F}$ $\mathcal{G}$ σ -algebra

Def. $P(A|\mathcal{G})$ è quella v.o. $\mathcal{G}$ -misurabile a valori in $[0,1]$ t.c. $\forall B \in \mathcal{G}$

$$P(A \cap B) = \mathbb{E}[\mathbb{1}_{A \cap B}] = \mathbb{E}[\mathbb{1}_B \mathbb{1}_A] = \mathbb{E}[\mathbb{1}_B P(A|\mathcal{G})]$$

APPROFONDIMENTO *

È ben definita grazie al teorema di Radon-Nikodym

- Due misure P e Q su $(\Omega, \mathcal{F})$ si dicono ^{che Q è} ass. continui ^{rispetto a P} $Q \ll P$ e $\forall A \in \mathcal{F} : P(A) = 0 \Rightarrow Q(A) = 0$

- P e Q si dicono equivalenti se $Q \ll P$ e $P \ll Q$

Teorema di Radon-Nikodym

Se $Q \ll P$ allora $\exists$ una v.o. L $L = \frac{dQ}{dP} \Big|_{\mathcal{F}}$

t.c. $Q(B) = \mathbb{E}[\mathbb{1}_B L]$ (media rispetto alla misura P)

Esisto $A \in \mathcal{F}$ definiamo Q su $\mathcal{G}$: $Q(B) = P(A \cap B)$

allora $Q \ll P$ su $\mathcal{G}$, vede se $P(B) = 0 \Rightarrow Q(B) = 0$

$\Rightarrow$ per il teorema di R-N $\exists L$ v.o. $\mathcal{G}$ -misurabile:

$$Q(B) = P(A \cap B) = \mathbb{E}[\mathbb{1}_B L]$$

$$L = P(A|\mathcal{G})$$

Approfondimenti

- Sia $\mathcal{B} = \sigma(\mathcal{B}) = \{B, B^c, \emptyset, \Omega\}$

$P(A|\mathcal{B})$ è costante su B e su B^c perché $\mathcal{B}$ -misurabile

$$\forall \omega \in B \quad \mathbb{E}[\mathbb{1}_B P(A|\mathcal{B})] = P(A|\mathcal{B})(\omega) P(B) = P(A \cap B)$$

$$\forall \omega \in B^c \quad \mathbb{E}[\mathbb{1}_{B^c} P(A|\mathcal{B})] = P(A|\mathcal{B})(\omega) P(B^c) = P(A \cap B^c)$$

$$\Rightarrow P(A|\mathcal{B})(\omega) = \begin{cases} \frac{P(A \cap B)}{P(B)} & \omega \in B \\ \frac{P(A \cap B^c)}{P(B^c)} = \frac{P(A|B)}{P(B)} & \omega \in B^c \end{cases}$$

- Sia $\mathcal{B} = \sigma(X)$ scriviamo $P(A|\mathcal{B}(X)) = P(A|X)$
 allora $\exists$ una funzione $g(x)$:
 $g(x) = P(A|X)$
 queste funzioni si indicano con le notazioni: $g(x) = P(A|X=x)$

- Sia $\mathcal{B} = \sigma(X)$ e $A = (Y \in I)$ $P(Y \in I | X)$
 è quella u.a. $\sigma(X)$ -misur. t.c. $\forall B \in \sigma(X)$ $B = \{X \in I\}$

$$P(Y \in I, X \in I) = \mathbb{E}[\underbrace{\mathbb{1}_{(X \in I)}}_{H(X)} P(Y \in I | X)]$$

$$= \int_{\mathbb{R}} H(x) f_X(x) dx = \int P(Y \in I | X=x) f_X(x) dx$$

Di altra parte $P(Y \in I, X \in I) = \iint_{I \times I} f_{(X,Y)}(x,y) dx dy$

Dunque abbiamo che:

$$\forall I \in \mathcal{B}(\mathbb{R}) \quad \int_I P(Y \in J | X=x) f_X(x) dx = \int_I \int_J f_{(X,Y)}(x,y) dy dx$$

$$\Rightarrow P(Y \in J | X=x) = \int_J \frac{f_{(X,Y)}(x,y)}{f_X(x)} dy = \int_J f_{Y|X}(y|x) dy$$

$$\Rightarrow P(Y \in J | \sigma(X)) = P(Y \in J | X) = \int_J f_{Y|X}(y|x) dy$$

Media condizionata ad una σ -algebra

$$(\mathcal{X}, \mathcal{F}, P)$$

$$\mathcal{B} \subseteq \mathcal{F}$$

sub- σ -algebra

Def. X v.a.

$E(X | \mathcal{B})$ v.a. $\mathcal{B}$ -misurante tale che

$$\forall B \in \mathcal{B} \quad E(\mathbb{1}_B X) = E(\mathbb{1}_B E(X | \mathcal{B}))$$

In particolare prendendo $X = \mathbb{1}_A = \begin{cases} 1 & \text{su } A \\ 0 & \text{su } A^c \end{cases}$

$E(\mathbb{1}_A | \mathcal{B})$ è quella v.a. $\mathcal{B}$ -misurante t.c.

$$\forall B \in \mathcal{B} \quad E(\mathbb{1}_B \mathbb{1}_A) = E(\mathbb{1}_B E(X | \mathcal{B})) = E(\mathbb{1}_B E(\mathbb{1}_A | \mathcal{B}))$$

D'altra parte $P(A | \mathcal{B})$ è una s.c. $\mathcal{B}$ -mis. t.c.

$$\forall B \in \mathcal{B}$$

$$P(A \cap B) = E(\mathbb{1}_B P(A | \mathcal{B}))$$

$$\Rightarrow E(\mathbb{1}_B E(\mathbb{1}_A | \mathcal{B})) = E(\mathbb{1}_B P(A | \mathcal{B})) \quad \forall B \Rightarrow E(\mathbb{1}_A | \mathcal{B}) = P(A | \mathcal{B})$$

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4) Se $X \perp B$ (X è indipendente da B)

$$E(X|B) = E(X)$$

Dimostrare questo risultato per $X = I_A$ $A \in \mathcal{F}$

$$\forall B \in \mathcal{G} \quad P(A \cap B) = E[I_B E(I_A|B)]$$

" per l'indipendenza

$$P(A)P(B)$$

$$E(I_A)P(B) = E[I_B E(I_A)]$$

$\Rightarrow$ essendo entrambe v.c. $\mathcal{G}$ -mis.
 $E(I_A|B) = E(I_A)$

$$P(A|B) = P(A)$$

Per la circolarità del valore atteso condizionale, prop. 3) otteniamo da Eq 4) vale anche per le v.c. semplici $X = \sum_{i=1}^n \alpha_i I_{A_i}$

oppure direttamente:

$$\forall B \in \mathcal{G} \quad E(I_B X) = E(I_B E(X|B))$$

per l'indip.

$$E(I_B)E(X)$$

"

$$E(I_B E(X))$$

Per l'es. 2c) del foglio n.1

$$\Rightarrow E(X|B) = E(X)$$

essendo entrambe v.c. $\mathcal{G}$ -mis.

(5) Monotonic $X \leq Y$ quasi certamente $\Rightarrow E(X|\mathcal{B}) \leq E(Y|\mathcal{B})$
(senza dimostrazione)

(6) Se $Y \in \mathcal{B}$ -misurabile $E(XY|\mathcal{B}) = Y E(X|\mathcal{B})$

$$\forall B \in \mathcal{B} \quad E(1_B X Y) = E(1_B Y E(X|\mathcal{B}))$$

$$X Y = 1_C \quad C \in \mathcal{B}$$

$$E(1_B 1_C X) = E(1_{B \cap C} X) = E(1_{B \cap C} E(X|\mathcal{B})) = E(1_B 1_C E(X|\mathcal{B}))$$

$$B \cap C \in \mathcal{B}$$

$$\Rightarrow E(X 1_C | \mathcal{B}) = 1_C E(X|\mathcal{B}) \quad \text{per } Y = 1_C$$

~~Proprietà~~

(7) Proprietà delle torri e dei condizionamenti successivi:

$$\mathcal{B}_1 \subseteq \mathcal{B}_2 \quad E(E(X|\mathcal{B}_2)|\mathcal{B}_1) = E(E(X|\mathcal{B}_1)|\mathcal{B}_1) = E(X|\mathcal{B}_1)$$

• $E(E(X|\mathcal{B}_2)|\mathcal{B}_1) = E(X|\mathcal{B}_1)$ segue dalle 2) in quanto $E(X|\mathcal{B}_1)$ è anche $\mathcal{B}_2$ -misurabile
Rimane da mostrare che

$$E(E(X|\mathcal{B}_2)|\mathcal{B}_1) = E(X|\mathcal{B}_1)$$

$\forall B \in \mathcal{B}_1$ per def. di media condizionata a $\mathcal{B}_1$

$$E(1_B E(X|\mathcal{B}_2)) = E(1_B E(E(X|\mathcal{B}_2)|\mathcal{B}_1))$$

data parte $B \in \mathcal{B}_1 \subseteq \mathcal{B}_2$

$$\mathbb{E}(\mathbb{1}_B \mathbb{E}(X | \mathcal{B}_2)) \stackrel{\downarrow}{=} \mathbb{E}(\mathbb{1}_B X) \stackrel{\uparrow}{=} \mathbb{E}(\mathbb{1}_B \mathbb{E}(X | \mathcal{B}_1))$$

$B \in \mathcal{B}_1$

$$\Rightarrow \mathbb{E}(\mathbb{1}_B \mathbb{E}(\mathbb{E}(X | \mathcal{B}_2) | \mathcal{B}_1)) = \mathbb{E}(\mathbb{1}_B \mathbb{E}(X | \mathcal{B}_1)) \quad \forall B \in \mathcal{B}_1$$

$\Downarrow$

$$\mathbb{E}(\mathbb{E}(X | \mathcal{B}_2) | \mathcal{B}_1) = \mathbb{E}(X | \mathcal{B}_1)$$

8) Disuguaglianza di Jensen: $\varphi(x)$ funzione convessa

$$\mathbb{E}[\varphi(X) | \mathcal{B}] \geq \varphi(\mathbb{E}(X | \mathcal{B}))$$

Esempio: $\varphi(x) = x^2$

$$(*) \quad \mathbb{E}(X^2 | \mathcal{B}) \geq \mathbb{E}(X | \mathcal{B})^2 \quad \Rightarrow \quad |\mathbb{E}(X | \mathcal{B})| \leq \sqrt{\mathbb{E}(X^2 | \mathcal{B})}$$

• $\varphi(x) = e^x$

$$\mathbb{E}(e^X | \mathcal{B}) \geq e^{\mathbb{E}(X | \mathcal{B})}$$

Inoltre prendendo il valore atteso nella (*)

$$\mathbb{E}(\mathbb{E}(X^2 | \mathcal{B})) = \mathbb{E}(X^2) \geq \mathbb{E}(\mathbb{E}(X | \mathcal{B})^2)$$

$$\Rightarrow \mathbb{E}(\mathbb{E}(X | \mathcal{B})^2) \leq \mathbb{E}(X^2)$$